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Interaction algebras of spin models and their relevance for disorder solutions

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Abstract. The interaction algebra of a discrete classical spin model with transitive symmetry group is generated by the s independent matrices corresponding to its set of minimal edge-transitive graphs. If this algebra is non-Abelian, its centre is at least the Potts model interaction algebra. For a spin model on the square lattice with four different interactions, the existence of an Abelian interaction algebra implies the existence of four $3s$-dimensional disorder solutions. In the non-Abelian case, there are at least four $(2s+2)$-dimensional disorder solutions. As examples, the Potts and Ashkin–Teller models (Abelian) and the $F(6)$-model (non-Abelian) are discussed. The possibility of an expansion around a disorder solution in terms of a class of subgraphs of the shadow lattice is indicated. Inversion relations for spin models on the checker-board lattice are derived using an explicit form for the inverse of the transfer matrix.

1. Introduction

The simplest way to consider a statistical-mechanical model on a graph or lattice is obtained by assuming that all the interactions are equal. If this requirement is relaxed, the problem quickly becomes intractable, unless the number of different interactions is kept small. This is, for example, the case for the triangular and square lattices, which are such that their duals are bipartite. It is, therefore, possible to select half of the faces of these lattices and define three (triangular case) or four (square case) different interactions around each face. In figure 1(a) this basic face for the square lattice is shown, together with the numeration of the four interactions adopted in this article. Figure 1(b) shows the basic face of the triangular lattice; this may be considered as a special case of the square lattice with interaction 3 (figure 1(a)) replaced by the unit matrix (the limit of infinitely strong attraction). Therefore, only the square-lattice case (also called the checker-board lattice with these interactions) is considered in what follows.

For the Potts model on the triangular lattice, it has been shown [1], that there are three two-dimensional disorder solutions in the three-dimensional thermodynamic state space. For such a disorder solution, all triangles are frustrated [2], so that the partition function has a very simple form. This result has been generalized to the square lattice [3], in which there are four three-dimensional disorder solutions in the four-dimensional state space. Disorder solutions have also been found for other types of models, see, e.g., [4–8] for the earliest references. Such disorder solutions have been found using a variety of methods, of which the 'crystal-growth' method is the most prominent. In the present article, it is shown that for discrete classical spin models, the actual determination of the disorder solutions is almost trivial.

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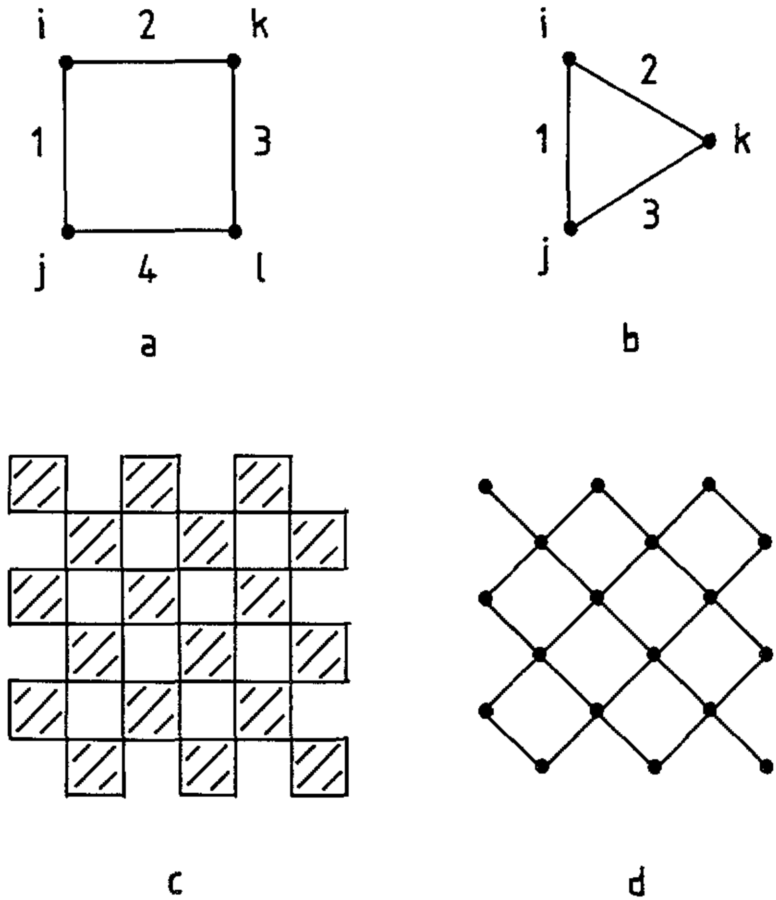

Figure 1. (*a*) Basic square with four interactions of the checker-board lattice. (*b*) Basic triangle with three interactions of the triangular lattice. (*c*) Part of the checker-board lattice with the hatched squares as in (*a*). (*d*) Part of the shadow lattice corresponding to (*c*).

A general spin model with transitive symmetry group G and M states will have s independent energy parameters, or Boltzmann factors, corresponding to the s minimal edge-transitive graphs associated with G (see also section 2), so that the thermodynamic state space for such a model on the checker-board lattice is $4s$-dimensional. The present article answers the question as to the existence and dimensionality of disorder solutions in this general case as well as some related questions. The paper is organized as follows: in section 2, general discrete classical spin models are discussed. The interaction algebra for such a model is defined and the conditions under which this algebra is commutative (Abelian) are given. The centres of non-Abelian interaction algebras are discussed. In section 3, the basic requirement for the existence of a disorder solution is derived. It is shown that there are four $3s$-dimensional disorder solutions in the state space in the Abelian case, whereas there are again at least four $(2s+2)$-dimensional order solutions in the non-commutative case. As examples, the groups $S(M)$ (M-state Potts model), $S(2) \otimes S(2)$ (Ashkin–Teller model) and $F(6)$ (the model with the minimal number of states having a non-Abelian interaction algebra) are discussed. Since the disorder solutions are generalizations of the high-temperature limit, an expansion around a disorder solution is akin to a high-temperature expansion. Section 4 is devoted to a short discussion of expansions around disorder solutions in terms of connected section graphs of the shadow lattice. In section 5, a discussion of the 'inversion relations' [3,9–12], which have been conjectured to exist for the Potts model on the checker-board lattice, is given for the general case.

2. Spin models and their interaction algebras

A classical discrete spin model with M possible spin states is completely defined by its permissible symmetry group G. Its interaction matrix $E(i,j)$, which is the energy associated with an (unordered) pair of interacting spins in states i and j, is given by ($E(i,i)=0$ for all i can be chosen by the transitivity of G)

$$E(i,j)=\sum_{k=1}^{s} E_k M_k(i,j) \tag{2.1}$$

where the E_k are arbitrary parameters, each associated with a symmetric matrix $M_k(i,j)$. $M_k(i,j)$ is the incidence matrix of the graph L_k obtained from a single (undirected) edge by the permutations of the group G. This maximal set of symmetric G-invariant graphs is called the maximal interaction of G. If $G(L_k)$ is the automorphism group of the graph L_k, then G equals the intersection of all groups of this type

$$G=\bigcap_{k=1}^{s} G(L_k). \tag{2.2}$$

This expresses the permissibility of G. For more details on these matters, see the book [13]; extensions to models with infinitely many states are discussed in [14].

The algebra $I(G)$, generated by the M_k-matrices, (and the $M\times M$ unit matrix) is called the interaction algebra of G. It can be shown [13], that this algebra is Abelian if, and only if, the representation of G on the M states contains each real irreducible representation of G either once or not at all. This is certainly the case if G is completely permissible, which means that for every pair of different states i and j, there is an element g from G with $g(i)=j$, $g(j)=i$. Complete permissibility implies that all G-invariant matrices are necessarily symmetric. For $M\leqslant 10$, every spin model with Abelian interaction algebra is completely permissible; of the 41 spin models with no more than 10 states, only four have non-Abelian interaction algebras [13].

Since the graphs L_k are regular, i.e. have equal valency z_k (valency of a vertex equals the number of edges emanating from it) at every vertex, the $M\times M$ matrix $Q(i,j)$, with all entries equal to 1, commutes with all M_k-matrices:

$$QM_k=M_kQ=z_kQ. \tag{2.3}$$

One also has

$$I+\sum_{k=1}^{s} M_k=Q \tag{2.4}$$

where I is the $M \times M$ unit matrix, so that $Q \in I(G)$. The set of matrices in $I(G)$ which commute with all matrices from this algebra, called the *centre* of $I(G)$, therefore contains all matrices of the type

$$A_{\mathrm{p}}(i,j) = (aI + bQ)(i,j) = \begin{cases} a+b & \text{for } i = j \\ b & \text{for } i \neq j. \end{cases} \tag{2.5}$$

In particular, the centre of $I(G)$ contains the Boltzmann-factor matrices of the Potts model:

$$\Omega_{\mathrm{p}}(i,j) = \begin{cases} 1 & \text{for } i = j \\ \omega & \text{for } i \neq j. \end{cases} \tag{2.6}$$

The general Boltzmann-factor matrix corresponding to equation (2.1) is simply given by

$$\Omega(i,j) = \exp(-\beta E(i,j)). \tag{2.7a}$$

Property (2.4) and the nature of the $M_k(i,j)$ as incidence matrices of edge-disjunct graphs imply the explicit form:

$$\Omega(i,j) = \sum_{k=1}^{s} \omega_k M_k(i,j) + \delta(i,j) \qquad \omega_k = \exp(-\beta E_k) \tag{2.7b}$$

from which equation (2.4) is obtained by setting all energy parameters (or all ω_k) equal. This shows that the algebra $I(S(M))$ is Abelian ($S(M)$ is the symmetry group of the Potts model).

If the spin-model group G contains a regular Abelian subgroup A, then G is completely permissible since either A consists only of involutions or the permissibility of G implies that G also contains the automorphism of A, which maps each element a of A onto its inverse a^{-1}. In both cases, G contains for all i, j elements g with $g(i) = j$, $g(j) = i$. In particular, if $A = C(M)$, the cyclic group on M objects, then G contains $D(M)$, the dihedral group corresponding to $C(M)$, and the Boltzmann-factor matrix has the form

$$\Omega(i,j) = \omega_{|i-j|} \qquad \omega_0 = 1. \tag{2.8}$$

$I(D(M))$ is again commutative, since all cyclic matrices of the form (2.8) commute. A further example is afforded by the Ashkin–Teller model; with group $S(2) \otimes S(2)$ and Boltzmann-factor matrix

$$\Omega(\mathrm{AT}) = \begin{pmatrix} 1 & \omega_1 & \omega_2 & \omega_3 \\ \omega_1 & 1 & \omega_3 & \omega_2 \\ \omega_2 & \omega_3 & 1 & \omega_1 \\ \omega_3 & \omega_2 & \omega_1 & 1 \end{pmatrix}. \tag{2.9}$$

This is an example of the case in which the Abelian group consists of involutions only.

As examples of groups with non-Abelian interaction algebras, we now consider the groups $F(2N)$. These are isomorphic to the dihedral groups $D(N)$, but represented on $2N$ objects. They are given explicitly by [13]

$$F(2N) = \{e, g^2, g^4, \ldots, g^{2N-2}, \sigma g, \sigma g^3, \ldots, \sigma g^{2N-1}\} \tag{2.10}$$

where g is a generator of $C(2N)$, $g^{2N}=e$ and σ represents the automorphism of $C(2N)$ which maps onto inverses:

$$\sigma g^a \sigma = g^{-a} \qquad \sigma\sigma g^b \sigma = g^b \sigma = \sigma g^{-b}. \tag{2.11}$$

The maximal interaction of $F(2N)$ contains, in addition to other graphs, the graphs L_1 and L_2 given by

$$L_1 = \{(2n-1, 2n), n = 1, 2, \dots, N\} \qquad L_2 = \{(2n, 2n+1), n = 1, 2, \dots, N\} \tag{2.12}$$

which together make up the circle $\{(n, n+1), n = 1, \dots, 2N\}$ with group $D(2N)$, of which $F(2N)$ is an index-2 (normal) subgroup. The matrices M_1 and M_2, corresponding to L_1 and L_2, do not commute. For $N > 2$,

$$(M_2M_1)(i,j) = \begin{cases} 1 & \text{if } i \text{ is odd and } j = i+2 \\ 1 & \text{if } i \text{ is even and } j = i-2 \\ 0 & \text{otherwise} \end{cases} \tag{2.13a}$$

$$(M_1M_2)(i,j) = \begin{cases} 1 & \text{if } i \text{ is odd and } j = i-2 \\ 1 & \text{if } i \text{ is even and } j = i+2 \\ 0 & \text{otherwise.} \end{cases} \tag{2.13b}$$

The sum $M_1M_2 + M_2M_1$ is again a matrix corresponding to a graph from the maximal interaction of $F(2N)$.

For the case $N = 3$ ($M = 6$), the Boltzmann-factor matrix has the explicit form

$$\Omega = \begin{pmatrix} 1 & \omega_1 & \omega_4 & \omega_3 & \omega_4 & \omega_2 \\ \omega_1 & 1 & \omega_2 & \omega_4 & \omega_3 & \omega_4 \\ \omega_4 & \omega_2 & 1 & \omega_1 & \omega_4 & \omega_3 \\ \omega_3 & \omega_4 & \omega_1 & 1 & \omega_2 & \omega_4 \\ \omega_4 & \omega_3 & \omega_4 & \omega_2 & 1 & \omega_1 \\ \omega_2 & \omega_4 & \omega_3 & \omega_4 & \omega_1 & 1 \end{pmatrix} \tag{2.14}$$

i.e. it is a cyclic matrix, apart from the M_1 and M_2 parts. The structure of the interaction algebra follows as

$$\begin{aligned} &M_1M_2 + M_2M_1 = M_1M_3 + M_3M_1 = M_2M_3 + M_3M_2 = M_4 \\ &[M_1, M_2] \neq 0 \qquad [M_1, M_3] \neq 0 \qquad [M_2, M_3] \neq 0 \\ &M_iM_4 = M_4M_i = M_j + M_k \qquad i = 1, 2, 3 \end{aligned} \tag{2.15}$$

where j and k are the values from $(1 \to 3)$ unequal to i. Since M_4 commutes with all other matrices, the centre of $I(F(6))$ consists of all matrices of the form

$$Z = aI + bM_4 + c(Q - I - M_4). \tag{2.16}$$

This is exactly the interaction algebra $I(S(3) \wr S(2))$ of the group of the graph corresponding to M_4, which consists of two triangles, by equation (2.14).

3. Disorder solutions

We consider a finite portion of the checker-board lattice, as in figure 1(c), where the hatched squares are squares from figure 1(a). The partition function for free boundary conditions can be written symbolically as

$$Z = \sum_{ijkl} \prod_{\mathrm{sq}} M(i, j, k, l) \tag{3.1}$$

where the product is over all hatched squares and the sum over all vertex variables. The quantity $M(i, j, k, l)$ is given as (figure 1(a))

$$M(i, j, k, l) = \Omega^{(1)}(i, j)\Omega^{(2)}(i, k)\Omega^{(3)}(k, l)\Omega^{(4)}(l, j). \tag{3.2}$$

A disorder solution is said to exist for interactions which are such that the partition function can be evaluated by summing over the variables in 'layers'; for instance, if one starts at the extreme right, the condition

$$\sum_{k,l} M(i, j, k, l) = \text{independent of } i \text{ and } j \tag{3.3}$$

ensures that this process can be iterated. Now the sum over k and l simply performs a matrix product in equation (3.2) so that the condition of equation (3.3) implies

$$[\Omega^{(1)}(i, j)]^{-1} = f_0^{-1}(\Omega^{(2)}\Omega^{(3)}\Omega^{4)})(i, j). \tag{3.4}$$

Here f_0 follows from $\Omega^{(1)}(i, i) = 1$ as

$$f_0 = (\Omega^{(2)}\Omega^{(3)}\Omega^{(4)})(i, i) \tag{3.5}$$

which is independent of i by the transitivity of G. If a solution of equation (3.4) exists, then the partition function in the thermodynamic limit follows as

$$\lim_{V\to\infty} V^{-1} \ln Z = \tfrac{1}{2} \ln f_0. \tag{3.6}$$

By starting the summation in one of the (three) other directions, equations similar to (3.4) and (3.5) are obtained by a cyclic permutation of the superscripts ($1 \to 4$).

Obviously, equation (3.4) is (easily) solvable if, and only if, the matrix product $\Omega^{(2)}\Omega^{(3)}\Omega^{(4)}$ is again a symmetric matrix. If the group G has an Abelian interaction algebra, this is always the case. Moreover, the solutions for $\Omega^{(1)}$ and f_0 will be symmetric with respect to any permutation of the superscripts ($2 \to 4$). Therefore, one obtains the following: if $I(G)$ is Abelian, then there are four $3s$-dimensional subspaces of the $4s$-dimensional thermodynamic state space in which a disorder solution exists. Since the actual evaluation of the matrix product is nearly trivial, only two examples (which do not give too unwieldy expressions) will be given.

(i) For the M-state Potts model, the matrix elements of $\Omega^{(2)}\Omega^{(3)}\Omega^{(4)}$ are

$$1 + (M-1)(\omega^{(2)}\omega^{(3)} + \omega^{(2)}\omega^{(4)} + \omega^{(3)}\omega^{(4)}) + (M-1)(M-2)\omega^{(2)}\omega^{(3)}\omega^{(4)} \qquad i = j \tag{3.7a}$$

$$\omega^{(2)} + \omega^{(3)} + \omega^{(4)} + (M-2)(\omega^{(2)}\omega^{(3)} + \omega^{(2)}\omega^{(4)} + \omega^{(3)}\omega^{(4)}) + (M^2 - 3M + 3)\omega^{(2)}\omega^{(3)}\omega^{(4)} \qquad i \neq j. \tag{3.7b}$$

Then f_0 is given by equation (3.7a), whereas the off-diagonal elements of $\Omega^{(1)}$ are equal to the ratio of equations (3.7a) and (3.7b). This result (and the similar ones obtained by summing in the other directions) agrees with the conjecture made in [3]. By setting $\omega_3 = 0$, the triangular lattice result of [1] is recovered. It is easy to see that, for $\omega^{(2)}$, $\omega^{(3)}$ and $\omega^{(4)}$ all < 1 ('ferromagnetic' regime), $\omega^{(1)} > 1$ ('antiferromagnetic' coupling) follows, so that the squares are really frustrated on a disorder solution.

(ii) For the Ashkin–Teller model with Boltzmann-factor matrices, as in equation (2.9), one finds:

$$f_0 = 1 + \sum_{i=1}^{3} \sum_{(a,b)} \omega_i^{(a)} \omega_i^{(b)} + \sum_{(a,b,c)} \omega_1^{(a)} \omega_2^{(b)} \omega_3^{(c)} \tag{3.8}$$

where (a, b) denotes a pair of different indices from $(2 \to 4)$, whereas (a, b, c) is a permutation of $(2 \to 4)$. Further, the $\omega_j^{(1)}$ are given as

$$(\omega_j^{(1)})^{-1} f_0^{-1} = \sum_{a=2}^{4} \omega_j^{(a)} + \prod_{a=2}^{4} \omega_j^{(a)} + \sum_{(a,b)} \omega_k^{(a)} \omega_l^{(b)} + \tfrac{1}{2} \sum_{(a,b,c)} \omega_j^{(a)} (\omega_k^{(b)} \omega_l^{(c)} + \omega_l^{(b)} \omega_k^{(c)}) \tag{3.9}$$

where k and l are the values of $(1 \to 3)$ unequal to j. The compact form of equations (3.8) and (3.9) is due to the fact that the group $S(2) \otimes S(2)$ has $S(3)$ as an automorphism group, so that all expressions are invariant under permutations of $(1 \to 3)$.

For models with non-Abelian $I(G)$, equation (3.4) can only be solved if the matrix product is symmetric, i.e. if

$$(\Omega^{(4)})^{-1} \Omega^{(2)} \Omega^{(3)} \Omega^{(4)} (\Omega^{(2)})^{-1} = \Omega^{(3)} \tag{3.10}$$

holds. For a generic $\Omega^{(3)}$, this implies that

$$\Omega^{(4)} = (a\Omega^{(3)} + Z)\Omega^{(2)} \tag{3.11}$$

must hold, where Z is an element from the centre of $I(G)$. Since this centre contains (section 2) at least all M-state Potts-model matrices Ω_{p}, solvability is guaranteed if

$$\Omega^{(4)} = c(a\Omega^{(3)} + b\Omega_{\mathrm{p}})\Omega^{(2)} \tag{3.12}$$

holds, where c has to be chosen so that $\Omega^{(4)}(i, i) = 1$ is preserved. Since the Potts-model matrix still contains an arbitrary parameter, equation (3.12) implies that the disorder solution has dimension of at least $2s + 2$, since two Boltzmann-factor matrices and three parameters are restricted by only one normalization condition. In the special case of $F(6)$, for which the centre of the interaction algebra has three dimensions (being equal to $(I(S(3) \wr S(2)$, section 2), the disorder solutions have dimension 11 (in 16-dimensional state space) since equation (3.12) is here replaced by

$$\Omega^{(4)} = c(a\Omega^{(3)} + b_1 I + b_2 M_4 + b_3(Q - I - M_4))\Omega^{(2)} \tag{3.13}$$

with four parameters restricted by one normalization. An explicit form of the disorder solutions for this case will not be given, since it is extremely unwieldy and trivial to derive.

4. Expansions around disorder solutions

The disorder solutions derived in the previous section are analogous to the high-temperature limit, since there equation (3.3) for the $M(i, j, k, l)$ of equation (3.2) is also satisfied. An expansion around a disorder solution is, therefore, akin to a high-temperature expansion, see e.g., [14]. In the present case, however, some care must be taken, since there are different values of the partition function (given by equations (3.5) and (3.6)) on different portions of the disorder solution. The expansion is therefore restricted to that subspace of the state where

$$\sum_{i,j,k,l} M(i, j, k, l) = M^2 f_0 = \mathrm{Tr}(\Omega^{(1)}\Omega^{(2)}\Omega^{(3)}\Omega^{(4)}) \tag{4.1}$$

holds for a fixed value of f_0. For $f_0 > 1$, there is always a disorder solution with this f_0, as follows from equation (3.5). Now in this subspace, $M(i, j, k, l)$ can be written as

$$M(i, j, k, l) = f_0(1 + g(i, j, k, l))/M^2. \tag{4.2}$$

Insertion of this into equation (3.1) for the partition function gives:

$$Z = (f_0/M^2)^n \sum_{i,j,k,l} \prod_{\mathrm{sq}} (1 + g(i, j, k, l)) \tag{4.3}$$

where n is the number of hatched squares. By the transitivity of the group G of the spin model, one has

$$M^2 f_0 = \sum_{i,j,k,l} M(i, j, k, l) = M \sum_{a,b,c} M(i, j, k, l) = M^2 f_0 + (f_0/M) \sum_{a,b,c} g(i, j, k, l). \tag{4.4}$$

Here a, b, c are any three of the indices i, j, k, l, so that

$$\sum_{a,b,c} g(i, j, k, l) = 0. \tag{4.5}$$

Therefore, non-zero terms in an expansion of equation (4.3) must be such that none of the g-functions occurring has more than two free vertices. In case one is exactly on a disorder solution, the g-functions would have to have, at most, one free vertex, which is not possible for free boundary conditions, so here equation (4.3) reduces to equation (3.6). This shows once more that the form of equation (4.3) leads to an expansion around a disorder solution.

The non-zero terms in an expansion of equation (4.3) can be identified with subgraphs of the shadow lattice, shown in figure 1(d) for the part of the checker-board lattice shown in figure 1(c). This shadow lattice consists of a vertex for each hatched square; two vertices are connected by an edge if, and only if, the corresponding squares have a spin (vertex) in common. A non-zero contribution to an expansion of equation (4.3) then consists of a number of connected subgraphs of the shadow lattice with the following properties:

(i) each vertex has valency of at least two; and

(ii) if two neighbouring vertices of the shadow lattice belong to a graph, then so does the edge connecting them.

These requirements define the class of section graphs† of the square lattice (ii) without dangling ends (i). The first few connected graphs of this type are listed in figure 2. Standard

† The shadow lattice and section graphs are defined in [15].

methods (e.g. [14]) can then be also used to write the logarithm of the partition function as a series in these graphs. The evaluation of the contributions of these graphs is much more involved than in the high-temperature case [14]; the contribution of the order-4 graph of figure 2 is, for instance, in case there is an Abelian interaction algebra, given by

$$\begin{aligned} p(4) &= \mathrm{Tr}(R^{(1)}R^{(2)}R^{(3)}R^{(4)}) \\ 1 + M^{-2}R^{(i)}(a,b) &= \Omega^{(i)}(a,b)(\Omega^{(j)}\Omega^{(k)}\Omega^{(l)})(a,b)f_0^{-1}. \end{aligned} \tag{4.6}$$

This will not be pursued further here, but reserved for a future publication.

5. The transfer matrix and its inverse

In section 3, the disorder solutions are defined by the requirement that a simple iteration of the summation over the spin states in layers is possible. Away from a disorder solution, such a summation in layers can formally be achieved by the introduction of a transfer matrix connecting different layers. For the checker-board model on a cylinder, such a transfer matrix is easily defined: if there are $2l$ squares around the cylinder, then the matrix

$$T(i_1,\ldots,i_{2l};j_1,\ldots,j_{2l}) = \sum_{a_1,\ldots,a_{2l}} \prod_{t=1}^{l} M(i_{2t-1},i_{2t},a_{2t-1},a_{2t})M(a_{2t},a_{2t+1},j_{2t},j_{2t+1}) \tag{5.1}$$

(indices taken modulo $2l$) is such that if there are $2k$ squares in the 'horizontal' direction, then the partition function is

$$Z(k,l) = e \cdot T^k \cdot e \tag{5.2}$$

where the vector e represents free boundary conditions:

$$e(i_1,\ldots,i_{2l}) = 1 \qquad \text{for all } i_1,\ldots,i_{2l}. \tag{5.3}$$

On the disorder solution of equation (3.4), equation (3.3) implies that e is an eigenvector of T and equation (5.2) reduces to equation (3.6) in the thermodynamic limit.

The structure of equation (5.1) makes it easy to construct the inverse of the transfer matrix: its transpose $(T^{-1})^{\mathrm{t}}$ has the same structure as equation (5.1), but with $M(i,j,k,l)$ replaced by its transposed inverse:

$$[T(M)^{-1}]^{\mathrm{t}} = T((M^{-1})^{\mathrm{t}}). \tag{5.4}$$

Now this inverse of $M(i,j,k,l)$ is easily seen to be given by

$$M^{-1}(i,j,k,l) = \Omega^{(2)^{-1}}(i,k)\Omega^{(4)^{-1}}(j,l)/[\Omega^{(1)}(k,l)\Omega^{(3)}(i,j)] \tag{5.5}$$

where $\Omega^{-1}(i,j)$ is the matrix inverse of $\Omega(i,j)$. Since the matrix inverse has the same structure (symmetry group) as the original matrix, an inverse Boltzmann matrix can be defined by normalization:

$$\bar{\Omega}(i,j) = \Omega^{-1}(i,j)/\Omega^{-1}(i,i). \tag{5.6}$$

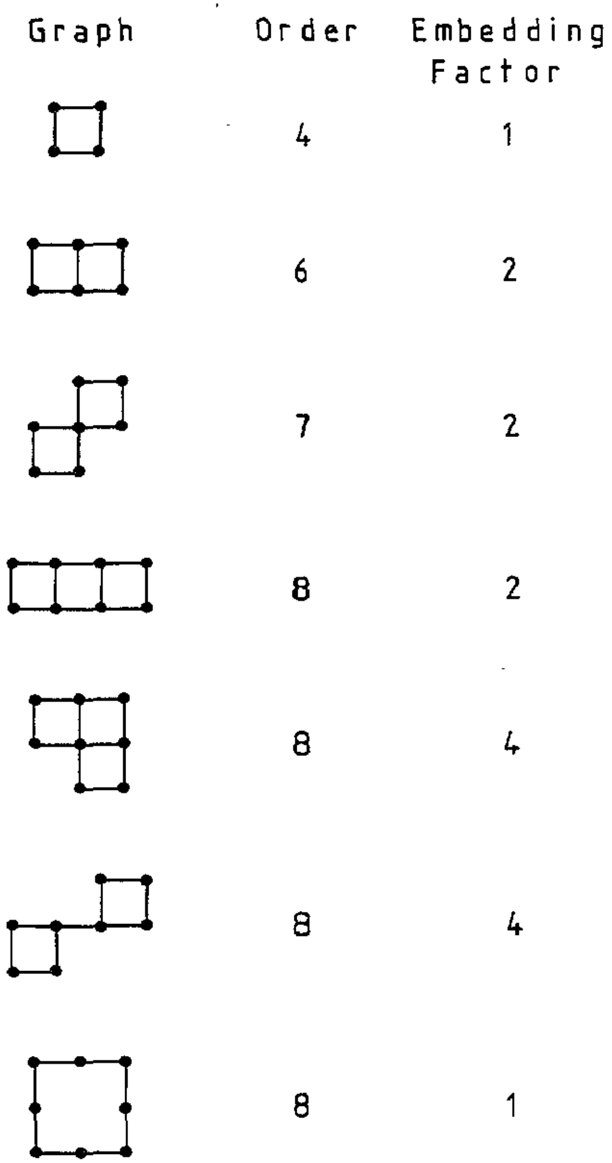

Figure 2. The subgraphs of the shadow lattice contributing to an expansion around a disorder solution for order < 9. Indicated are the order in the expansion (equal to the number of vertices) and the embedding factors.

(This matrix will, in general, contain non-physical values of the individual Boltzmann factors, see below.) With definition (5.6), equation (5.5) can be written as

$$M^{-1}(\Omega^{(1)}, \Omega^{(2)}, \Omega^{(3)}, \Omega^{(4)}) = g_0 M(1/\Omega^{(3)}, \bar{\Omega}^{(2)}, 1/\Omega^{(1)}, \bar{\Omega}^{(4)}) \tag{5.7a}$$

$$g_0 = \Omega^{(2)^{-1}}(i,i)\Omega^{(4)^{-1}}(i,i) \qquad (1/\Omega)(i,j) = 1/\Omega(i,j). \tag{5.7b}$$

For the Potts model, to which we restrict ourselves in the following, equation (5.6) is explicitly given by

$$\bar{\Omega}(i,j) = \begin{cases} 1 & \text{for } i = j \\ -\omega/[1+(M-2)\omega] = \bar{\omega} & \text{for } i \neq j \end{cases} \tag{5.8a}$$

$$\Omega^{-1}(i,i) = [1+(M-2)\omega]/[(1-\omega)(1+(M-1)\omega)] = h(\omega). \tag{5.8b}$$

This shows that the result gives unphysical (negative) values of the Boltzmann factors. With equations (5.8), equations (5.7) can be written as:

$$M^{-1}(\omega^{(1)}, \omega^{(2)}, \omega^{(3)}, \omega^{(4)}) = g_0 M(1/\omega^{(3)}, \bar{\omega}^{(2)}, 1/\omega^{(1)}, \bar{\omega}^{(4)}) \tag{5.9a}$$

$$g_0 = h(\omega^{(2)})h(\omega^{(4)}). \tag{5.9b}$$

The authors of [3, 9, 10] proposed the existence of an 'inversion relation' for the checkerboard Potts model of the form

$$Z(\omega^{(1)}, \omega^{(2)}, \omega^{(3)}, \omega^{(4)}) Z(1/\omega^{(3)}, \bar{\omega}^{(2)}, 1/\omega^{(1)}, \bar{\omega}^{(4)}) = g_0^{kl}(e \cdot e)^2 \tag{5.10}$$

where the second factor is an analytic continuation of the first factor to unphysical values of the Boltzmann factors. The above derivation of T^{-1} and equation (5.2) show that this analytic continuation can be identified with the naive definition of the second factor based on T^{-1} if, and only if,

$$(e \cdot T^k \cdot e)(e \cdot T^{-k} \cdot e) = (e \cdot e)^2 \tag{5.11}$$

is true. This is certainly the case if e is an eigenvector of T, i.e. on a disorder solution. Away from such a solution, however, the left-hand side of equation (5.11) is, for large k, proportional to

$$(\lambda_{\max}/\lambda_{\min})^k \tag{5.12}$$

where $\lambda_{\max}$ and $\lambda_{\min}$ are the eigenvalues of T with maximal and minimal modulus, respectively (in the space not orthogonal to e). Therefore, the inversion relation of equation (5.10) holds naively only on a disorder solution. It has been shown, however, that equation (5.10) holds in the sense of analytic continuation in the whole parameter space [3, 11, 12]. It would be interesting to extend the methods of these authors to check whether the inversion relations for other models can also be obtained from the inverse of the transfer matrix, as given by equations (5.7).

In conclusion, it has been shown how the dimension of the centre of the interaction algebra $I(G)$ of a spin model with permissible symmetry group G enters into the dimension of the disorder solutions in the thermodynamic state space of this model on the checkerboard lattice. Expansions around disorder solutions have been indicated in terms of a special class of subgraphs of the square lattice (as shadow lattice). It has been suggested that the inversion relations for general spin models can be obtained in a naive fashion by calculating the inverse of the transfer matrix, for which a general expression is available.

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